Math 250 – Sect.3.5 – Limits at Infinity and Asymptotes

In chapter 1 we investigated limits as x approached some finite value, c. Today we will investigate what happens with certain functions as x increases **without bound**.

-example- Consider the function $f(x)=\dfrac{4x+3}{2x-6}$. Investigate what happens to $f(x)$ as x increases or decreases without bound.

Numerically: Graphically:

This *end behavior* describes the location of the **horizontal asymptote** for the rational function.

Defn: The line $y = L$ is a **horizontal asymptote** of the graph of f if $\lim_{x\to\infty} f(x)=L$ or $\lim_{x\to-\infty} f(x)=L$

Theorem: If r is a positive rational number, and c is any real number, then $\lim_{x\to\infty}\dfrac{c}{x^r}=$ __________

-example- Find each limit.

a. $\lim_{x\to\infty}\dfrac{4}{x+3}$

b. $\lim_{x\to-\infty}\left(6-\dfrac{3}{x}\right)$

*Not all limits go to zero, as we saw in our first example. Consider a function of the form $f(x)=\dfrac{p(x)}{q(x)}$, where both $p(x)$ and $q(x)$ go to infinity as x approaches infinity. This result is called an **indeterminate** form. To solve this dilemma, we will use the *end behavior* functions for the numerator and denominator.

-example- Find each limit.

a. $\lim\limits_{x\to\infty}\dfrac{4x-1}{x+3}$

b. $\lim\limits_{x\to\infty}\dfrac{5+2x-x^2}{x^2+4}$

c. $\lim\limits_{x\to\infty}\dfrac{x^2+5x}{3x+3}$

d. $\lim\limits_{x\to\infty}\dfrac{4x-3}{x^2+3x+1}$

Summary for limits to $\pm\infty$ for rational functions of the form $y=\dfrac{p(x)}{q(x)}$:

1. If the degree of p(x) is LESS THAN the degree of q(x), then

2. If the degree of p(x) is GREATER THAN the degree of q(x), then

3. If the degree of p(x) is EQUAL TO the degree of q(x), then

-example- Find the horizontal asymptote(s) for each function:

a. $f(x)=\dfrac{3x-5}{4-2x}$

b. $f(x)=\dfrac{3x-2}{\sqrt{4x^2+1}}$

*One more interesting function: Investigate the graph of $f(x)=\dfrac{2x^2-3x+5}{x^2+1}$

Sketch:

-example- Trig limits to infinity

a. $\lim\limits_{x\to\infty}\cos x$

b. $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}$

c. $\lim\limits_{x\to\infty}\dfrac{\sin(1/x)}{(1/x)}$

APPLICATIONS: Limits to infinity, in real world situation, often are used to describe what happens as time goes on.

-example- A small business is producing a certain type of product. They compute that if they produce *x* units, the average cost per unit can be computed using $\overline{C} = 0.5 + \dfrac{5000}{x}$.

a. Find the average cost per unit if 100 units are produced.

b. Find the average cost per unit if 1000 units are produced.

c. Find $\lim_{x\to\infty} \overline{C}$. Explain what this represents.

-example- The state game commission introduces 30 elk into a new state park. The population *N* of the herd is modeled by $N = \dfrac{10(3+4t)}{1+0.1t}$, where *t* is the time in years.

a. Find the size of the herd after 5, 10, and 25 years.

b. Find $\lim_{t\to\infty} N(t)$, and explain what this represents.

c. The population of a herd of elk in a different park is modeled by $N = \dfrac{200}{1+4e^{-0.2t}}$. Find $\lim_{t\to\infty} N(t)$ for this herd, and explain what this represents.